Binomial partial difference ideals

C.M. Yuan[1]

[1] *Academy of systems science, CAS, China, {cmyuan}@mmrc.iss.ac.cn*

In [1], we present some basic concepts and properties of binomial difference ideals. A natural question is that what about the partial difference case? The main difference between the ordinary difference case and the partial difference case is that in partial difference case, the difference operators may have relations and the algorithms for the ordinary difference case can not be used directly.

Let F be a partial difference field, and $\sigma_1,\ldots,\sigma_m$ be m partial difference operators. F is said to be *inversive* if for any i such that $\sigma_i a \in F$ implies $a \in F$. Let Θ be the multiplicative closed set with unit of these m partial difference operators. Let $Y = \{y_1,\ldots,y_n\}$ be n difference indeterminates. Then, the elements of $F\{Y\} = F[\Theta y_j; j = 1,\ldots,n]$ are called *partial difference polynomials* over F in Y, and $F\{Y\}$ itself is called the *partial difference polynomial ring* over F in Y. A *partial difference polynomial ideal* I in $F\{Y\}$ is an ordinary algebraic ideal which is closed under transforming, i.e. $\sigma_i(I) \subset I$ for any i. If I also has the property that for any $1 \le i \le m$, $\sigma_i a \in I$ implies that $a \in \mathscr{I}$, it is called a *reflexive partial difference ideal*. And a prime partial difference ideal is a partial difference ideal which is prime as an ordinary algebraic polynomial ideal.

Since if F is not algebraically closed, Bentsen [2] show that one irreducible polynomial may has no solution in its extension field. In order to avoid this, F is assumed to be algebraically closed, inversive and of characteristic zero. We introduce the following useful notation. Let $x_1,\ldots,x_m$ be m algebraic indeterminates and $p = \sum c_{k_1,\ldots,k_m} \prod_{i=1}^m x_i^{k_i} \in \mathbf{Z}[x_1,\ldots,x_m]$. For a in any over field of F, denote

$$a^p = \prod (\sigma_i^{k_i} a)^{c_{k_1,\ldots,k_m}}.$$

For instance, if $m = 2$, then $a^{x_1^2+2x_2-1} = (\sigma_1^2 a)(\sigma_2 a)^2/a$. It is easy to check that for $p,q \in \mathbf{Z}[x_1,\ldots,x_m]$, and a,b in any over field of F, we have

$$a^{p+q} = a^p a^q, a^{pq} = (a^p)^q, (ab)^p = a^p b^p. \tag{1}$$

For $\mathbf{f} = (f_1,\ldots,f_n)^\tau \in \mathbf{Z}[x_1,\ldots,x_m]^n$, we define $Y^{\mathbf{f}} = \prod_{i=1}^n y_i^{f_i}$. $Y^{\mathbf{f}}$ is called a *Laurent partial difference monomial* in Y and $\mathbf{f}$ is called its *support*.

A *Laurent partial difference polynomial* over F in Y is an F-linear combination of Laurent partial difference monomials in Y. Clearly, the set of all Laurent partial difference polynomials form a commutative partial difference ring under the obvious sum, product, and the partial difference operators σ_i, where all Laurent partial difference monomials are invertible. We denote the partial difference ring of Laurent partial difference polynomials with coefficients in F by $F\{Y^{\pm}\}$.

A polynomial ideal I is called binomial if it is generated by polynomials with at most two terms. By a *Laurent partial difference binomial* in Y, we mean a partial difference polynomial with two terms, that is, $aY^{\mathbf{f}_1}+bY^{\mathbf{f}_2}$ where $a,b\in F^*=F\setminus\{0\}$ and $\mathbf{f}_1,\mathbf{f}_2\in\mathbf{Z}[x_1,\dots,x_m]^n$. A Laurent partial difference binomial of the following form is said to be in *normal form*

$$p=Y^{\mathbf{f}}-c_{\mathbf{f}} \tag{2}$$

A Laurent partial difference ideal is called binomial if it is generated by Laurent partial difference binomials. Then, we have

Lemma 1 *Let I be a Laurent binomial partial difference ideal and*

$$L(I):=\{\mathbf{f}\in\mathbf{Z}[x_1,\dots,x_m]^n\,|\,\exists c_{\mathbf{f}}\in F^*\, s.t.\, Y^{\mathbf{f}}-c_{\mathbf{f}}\in I\}. \tag{3}$$

Then $L(I)$ is a $\mathbf{Z}[x_1,\dots,x_m]$ modular, which is called the support modular *of I.*

This lemma means that the properties of a Laurent binomial partial difference ideal is related to its support modular.

Based on this lemma and the partial characteristic set method [3], we may decompose the zero set of a Laurent binomial partial difference ideal I into a set of prime ideals with their characteristic set are strong irreducible. Moreover, let $\mathbf{m}$ be the monomial set in $x_1,\dots,x_m$, if we denote by $L_S=\{\mathbf{f}\in\mathbf{Z}[x_1,\dots,x_m]^n|\exists c\in\mathbf{Z},\mathbf{g}\in\mathbf{m}, s.t. cg\in L\}$.

Theorem 1 *Let I be a Laurent binomial partial difference ideal, L its support modular, and L_S the saturation of L. Then $\{I\}$ is either $[1]$ or can be written as the intersection of Laurent reflexive prime binomial partial difference ideals whose support modular is L_S.*

Definition 1 *Let L be a $\mathbf{Z}[x_1,\dots,x_m]$-module in $\mathbf{Z}[x_1,\dots,x_m]^n$.*

- *L is called* $\mathbf{Z}$-saturated *if, for any $a\in\mathbf{Z}$ and $\mathbf{f}\in\mathbf{Z}[x_1,\dots,x_m]^n$, $a\mathbf{f}\in L$ implies $\mathbf{f}\in L$.*
- *L is called* x-saturated *if, for any $\mathbf{f}\in\mathbf{Z}[x_1,\dots,x_m]^n$, $x_i\mathbf{f}\in L$ implies $\mathbf{f}\in L$ for any i.*
- *L is called* saturated *if it is both* $\mathbf{Z}$- *and x- saturated.*

Then, we have the following result

Theorem 2 *If F is algebraically closed and inversive, I is a non-trivial Laurent binomial partial difference ideal and L its support modular, then*

(a) *L is $\mathbf{Z}$-saturated if and only if I is prime;*

(b) *L is x-saturated if and only if I is reflexive;*

(c) *L is saturated if and only if I is reflexive prime.*

Also, we have algorithms to check whether a given $\mathbf{Z}[x_1,\ldots,x_m]$ modular L is $\mathbf{Z}$-saturated(x-saturated) or not, and in the negative case, to compute the $\mathbf{Z}$-saturation(x-saturation) of L.

Using these algorithms, we can decompose the zero set of a binomial partial difference ideal I into the union of the zero set of reflexive prime binomial partial difference ideals.

References

[1] X.S. Gao, Z. Huang, C.M. Yuan, *Binomial Difference Ideals*, arXiv:1603.03987, (2016).

[2] I. Bentsen, *The Existence of Solutions of Abstract Partial Difference Polynomials*, Trans- actions of the Amer. Math. Soc., 158, 373-397(1972).

[3] G.L. Zhang, X.S. Gao, *Properties of Ascending Chians fr Partial Difference Polynomial Systems*, LNAI 5081, 307-321 (2008).