Bases for Modules of Difference-Operators by Gröbner Reduction

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At the ISSAC 2015 conference, the authors have introduced the notion of **Gröbner Reduction** as an axiomatic approach to Gröbner bases techniques of filtered free modules [1]. Let R be a ring containing a field K, let F be a free R-module such that N is a submodule of F. The reduction relation $\longrightarrow$ is called a *Gröbner Reduction for N* provided that

1. $f \longrightarrow h \Rightarrow f \equiv h \pmod{N}$;
2. $\longrightarrow$ is a noetherian relation, i.e. every reduction sequence terminates in a finite number of steps;
3. The irreducible elements I in F build a monomial K-linear subspace, i.e. for all $f \in F$ we have $f \in I \Rightarrow \mathrm{supp}(f) \subseteq I$;
4. $I \cap N = 0$, i.e. every non-zero element is reducible;
5. For all $r \in \mathbf{N}^p$:
$$f \in F_r \wedge f \longrightarrow h \Rightarrow h \in F_r.$$

In that general setting, given a filtered module M_r (derived from a filtration R_r of the ring R), it is possible to compute the dimension of M_r viewn as K-vector space. In [2] it is shown that related ideas have been considered in literature for modules over non-commutative rings. The interplay of modules of this type (e.g. it is possible to model a differential ring by a particular Ore-algebra), confirm that a common theory might be applicable to modules over certain non-commutative rings. From that, it is possible to review

- **Reduction with Respect to Several Term Orderings** [3] for modules over the ring of Ore-polynomials;
- **Relative Reduction** [4] for modules over difference-differential rings;
- (x, ∂)**-Reduction** [5] for modules over Weyl-algebras,

under the aspect of Gröbner Reduction.

Recently, the authors have tried to view reduction in a more abstract setting. Precisely, if we have an ordered set $(Y, \leq)$ and an R-module M we consider a function $\mathrm{rk} : M \to Y$. We say that u reduces to v provided that

$$u \longrightarrow v :\Longleftrightarrow \exists g \in G \subseteq M \ \exists a \in A \subseteq R : v = u - ag \wedge \mathrm{rk}(v) < \mathrm{rk}(u). \quad (1)$$

Let now $N := RG$. A reduction defined by the data (G, A, rk)

- is noetherian;
- satisfies $f \longrightarrow h \Rightarrow f \equiv h \pmod N$.

We can show, that under this assumptions the following holds.

Theorem. Let (G, A, rk) define the reduction relation (1). Then

- $I \cap N \subseteq 0 \Leftrightarrow N = Z$ where $Z := \{f \in N : f \longrightarrow^{\star} 0\}$;
- $M = N + I$;
- The irreducibles I build a monomial K-vector space.

In this talk, we want to report on research results that we are currently developing, and give an idea how the idea of Gröbner bases over modules can be unified in a way that a common theory applies for computation of (vector-space) dimension for filtered modules over filtered rings.

References

[1] C. Fürst and G. Landsmann, *Computation of Dimension in Filtered Free Modules by Gröbner Reduction*, Proceedings of the International Symposium on Symbolic and Algebraic Computation 2015, ACM (ed.), pp. 181-188. 2015

[2] C. Fürst and G. Landsmann, *Three Examples of Gröbner Reduction over Noncommutative Rings.*, Technical report no. 15-16 in RISC Report Series, Research Institute for Symbolic Computation (RISC), Johannes Kepler University Linz, Austria. October 2015

[3] A. Levin, *Gröbner bases with Respect to Several Term Orderings and Multivariate Dimension Polynomials*, Proceedings of the International Symposium on Symbolic and Algebraic Computation 2007, ACM (ed.), pp. 251-260. 2007

[4] F. Winkler and M. Zhou, *Computing difference-differential dimension polynomials by relative Gröbner bases in difference-differential modules*, Journal of Symbolic Computation, 43/10, 726-745, 2008.

[5] C. Dönch and A. Levin, *Computation of Bivariate Characteristic Polynomials of Finitely Generated Modules over Weyl Algebras*, International Journal of Algebra and Computation, 23, no. 7 (2013) , 1625 - 1651.