On Computing Rational Generating Function of a Solution to the Cauchy Problem of Difference Equation

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Algorithms for computing rational generating functions of solutions of one-dimensional difference equations are well-known and easy to implement. We propose an algorithm for computing rational generating functions of solutions of two-dimensional difference equations in terms of initial data of the corresponding initial value problems. The crucial part of the algorithm is the reconstruction of infinite one-dimensional initial data on the basis of finite input data. The proposed technique can be used for the development of similar algorithms in higher dimensions. We furnish examples of implementation of the proposed algorithm.

The one-dimensional case is well-studied (see [1, 2]) due to the absence of geometric obstacles. In [3], A. Moivre considered the power series

$$f(0)+f(1)z+\ldots+f(k)z^k+\ldots$$

with the coefficients $f(0), f(1), \ldots$ satisfying the difference equation

$$c_m f(x+m)+c_{m-1}f(x+m-1)+\ldots+c_0 f(x)=0, \quad x=0,1,2\ldots, \tag{1}$$

where $c_m \neq 0$, and $c_j \in C$ are constants. He proved that this series always represents a rational function (De Moivre's Theorem, [3]).

In the multidimensional case, which is not adequately explored (see [4, 5, 6, 7]), the rational generating functions are the most useful class of generating functions according to the Stanley's hierarchy (see [8]). A broad class of two-dimensional sequences that lead to rational generating functions is well-known in the enumerative combinatorics. For example, one can consider the problem of finding a number of lattice paths, the problem on generating trees with marked labels, Bloom's strings, a number of placement of the pieces on the chessboard etc. (see [9, 10, 11]).

Generating functions for multiple sequences with elements which could be expressed in terms of rational, exponential functions or gamma function, form a wide class of hypergeometric-type functions [12, 13]. Their study leads to the problem of solving overdetermined systems of linear equations with polynomial coefficients.

A multidimensional analogue of the De Moivre Theorem was formulated and proved in [15]. We now give some definitions and notations that we will need for formulating the main result.

Let $x = (x_1, \ldots, x_n) \in \mathbf{Z}^n$, where $\mathbf{Z}^n = \mathbf{Z} \times \ldots \times \mathbf{Z}$ is the n-dimensional integer lattice. Let $A = \{\alpha\}$ be a finite set of points in $\mathbf{Z}^n$. Let $f(x)$ be a function of integer arguments $x = (x_1, \ldots, x_n)$ with constant coefficients c_α. By a difference equation with respect to the unknown function $f(x)$ we call the equation of the form

$$\sum_{\alpha \in A} c_\alpha f(x + \alpha) = 0. \tag{2}$$

In the present work we consider the case when the set A belongs to the positive octant $\mathbf{Z}_0^n = \{(x_1, \ldots, x_n), x_i \in \mathbf{Z}, x_i \geq 0, i = 1, \ldots, n\}$ of the integer lattice and satisfies to the condition:

There exists a point $m = (m_1, \ldots, m_n) \in A$ such that for any $\alpha \in A$ the inequalities

$$\alpha_j \leq m_j,\ j = 1, 2, \ldots, n \tag{3}$$

hold.

Denote the *characteristic polynomial* of (2) by

$$P(z) = \sum_{\alpha \in A} c_\alpha z^\alpha = \sum_{\alpha \in A} c_{\alpha_1, \ldots, \alpha_n} z_1^{\alpha_1} \cdots z_n^{\alpha_n}.$$

The generating function (or the z-transformation) of the function $f(x)$ in integer variables $x \in \mathbf{Z}_0^n$ is defined as follows:

$$F(z) = \sum_{x \geq 0} \frac{f(x)}{z^{x+I}}, \quad \text{where } I = (1, \ldots, 1).$$

Define the «initial data set» for the difference equation (2) that satisfies the condition (3), as follows:

$$X_0 = \{\tau \in \mathbf{Z}^n, \tau \geq 0,\ \tau \ngeq m\}.$$

Here $\ngeq$ means that the point τ belongs to the complement of the set defined by the system of inequalities

$$\tau_j \geq m_j,\ j = 1, \ldots, n.$$

The initial value problem is set up as follows: we need to find the solution $f(x)$ of the difference equation (2), which coincides with a given function $\varphi(x)$ on X_0:

$$f(x) = \varphi(x),\ x \in X_0. \tag{4}$$

It is easy to show (see, e.g., [6]), that if the condition (3) holds, then the initial value problem (2), (4) has the unique solution. The solvability of the problem (2), (4) without the constraints (3) has been studied in [4].

We now proceed with some more notations that we will need later on.

Let $J = (j_1, \dots, j_n)$, where $j_k \in \{0,1\}$, $k = 1, \dots, n$, be an ordered set. With each such set J we associate the face Γ_J of the n-dimensional integer parallelepiped $\Pi_m = \{x \in \mathbf{Z}^n : 0 \le x_k \le m_k, k = 1, \dots, n\}$ as follows:

$$\Gamma_J = \{x \in \Pi_m, x_k = m_k, \text{ if } j_k = 1, \text{and } x_k < m_k, \text{ if } j_k = 0\}. \tag{5}$$

For example, $\Gamma_{(1,\dots,1)} = \{m\}$ and $\Gamma_{(0,\dots,0)} = \{x \in \mathbf{Z}^n, 0 \le x_k < m_k,\ k = 1, \dots, n\}$.

It is easy to check that $\Pi_m = \bigcup\limits_J \Gamma_J$ and for any J, J' the corresponding faces do not intersect: $\Gamma_J \cap \Gamma_{J'} = \emptyset$.

Let $\Phi(z) = \sum\limits_{\tau \ge 0, \tau \ngeq m} \frac{\varphi(\tau)}{z^{\tau+I}}$ be the generating function of the initial data of the solution of (2), (4). With each point $\tau \in \Gamma_J$ we associate the series

$$\Phi_{\tau,J}(z) = \sum_{y \ge 0} \frac{\varphi(\tau + Jy)}{z^{\tau+Jy+I}},$$

and with each face Γ_J we associate the series

$$\Phi_J(z) = \sum_{\tau \in \Gamma_J} \Phi_{\tau,J}(z).$$

If we extend the domain of $\varphi(x)$ by zero on $\mathbf{Z}^n_+ \setminus X_0$, then the generating function of the initial data can be written down as the sum

$$\Phi(z) = \sum_J \Phi_J(z) = \sum_J \sum_{\tau \in \Gamma_J} \Phi_{\tau,J}(z).$$

Theorem 1. The generating function $F(z)$ of the solution of the problem (2),(4) under the assumption (3) and the generating function $\Phi(z)$ of the initial data are connected by the formula

$$P(z)F(z) = \sum_J \sum_{\tau \in \Gamma_J} \Phi_{\tau,J}(z) P_\tau(z), \tag{6}$$

where $P_\tau(z) = \sum\limits_{\alpha \le m, \alpha \nleq \tau} c_\alpha z^\alpha$ are polynomials.

For $n = 1$ it is easy to verify the statement of the Theorem 1.

For $n = 2$ Theorem 1 was proved in [14] in connection with studying the rational Riordan arrays. For $n > 1$ the proof was given in [15]. The properties of generating function of solutions of a difference equation in rational cones of integer lattice were studied by T. Nekrasova (see, e.g., [16]).

Theorem 1 yields the following multidimensional analog of the De Moivre Theorem that is essential for the construction of algorithm:

Theorem 2. The generating function $F(z)$ of the solution of the initial value problem (2), (4) under the assumption (3) is rational if and only if the generating function $\Phi(z)$ of the initial data is rational.

The proof of Theorem 2 was given in [15].

For $n = 1$ the expression for the generating function consists of a finite number of terms, which makes the corresponding algorithm and computational procedure trivial. In this case the input data consists of two finite sets of numbers, namely: coefficients of the difference equation and the initial data. The output data of the algorithm is a rational function.

In the case when $n > 1$, the initial data set X_0 is infinite. For $n = 2$ the algorithm for computing the generating function $F(z)$ can be reduced to computation of a finite number of one-dimensional generating functions of sequences with elements along the coordinate axes. These elements are uniquely determined by coefficients of the corresponding one-dimensional difference equation and by finite set of the corresponding initial data (that is different for each sequence).

Example. Bloom studies the number of singles in all 2^x x-length bit strings [11], where a single is any isolated 1 or 0, i.e. any run of length 1. Let $r(x,y)$ be the number of x-length bit strings beginning with 0 and having y singles. Apparently $r(x,y) = 0$ if $x < y$.

In [11] D. Bloom proves that $r(x,y)$ is a solution to the Cauchy problem

$$r(x+2,y+1) - r(x+1,y+1) - r(x+1,y) - r(x,y+1) + r(x,y) = 0$$

with the initial data $\varphi(0,0) = 1, \varphi(1,0) = 0, \varphi(x,0) = \varphi(x-1,0) + \varphi(x-2,0),\ x \geq 2, \varphi(0,y) = 0,\ y \geq 1, \varphi(1,1) = 1, \varphi(1,y) = 0,\ y \geq 2$.

The result is the generating function of the considered initial value problem:

$$F(z,w) = \frac{z-1}{z^2w - zw - w - z + 1}.$$

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