Difference Dimension Quasi-polynomials

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Let K be a (not necessarily inversive) difference field of zero characteristic with basic set of translations $\sigma = \{\alpha_1, \dots, \alpha_m\}$ that are assigned rational weights $w_1, \dots, w_m$, respectively. Let T denote the free commutative semigroup generated by σ and for any transform $\tau = \alpha_1^{k_1} \dots \alpha_m^{k_m} \in T$, let

$$\operatorname{ord}_w \tau = \sum_{i=1}^{m} w_i k_i.$$

Furthermore, for any $r \in \mathbb{N}$, let $T_w(r) = \{\tau \in T \mid \operatorname{ord}_w \tau \leq r\}$.

In this talk we present the following result.

Theorem. *With the above notation, let $L = K\langle \eta_1, \dots, \eta_n \rangle$ be a difference field extension of K generated by a finite set $\eta = \{\eta_1, \dots, \eta_n\}$ and for any $r \in \mathbb{N}$, let $L_r = K(\cup_{i=1}^{n} T_w(r)\eta_i)$. Then there exists a quasi-polynomial $\Phi_{\eta|K}(t)$ such that*

(i) $\Phi_{\eta|K}(r) = \operatorname{tr.deg}_K L_r$ *for all sufficiently large* $r \in \mathbb{N}$.

(ii) $\deg \Phi_{\eta|K} \leq m$.

(iii) $\Phi_{\eta|K}$ *is an alternative sum of Ehrhart quasi-polynomials associated with rational conic polytopes. The leading coefficient of the quasi-polynomial $\Phi_{\eta|K}$ is a constant that does not depend on the set of difference generators η of the extension L/K. Furthermore, the coefficient of t^m in $\Phi_{\eta|K}$ is equal to the difference transcendence degree of L/K.*

This theorem generalizes the corresponding result on difference dimension polynomials introduced in [3] (see also [4] where properties and applications of difference dimension polynomials are discussed). The quasi-polynomial $\Phi_{\eta|K}$ is called the *difference dimension quasi-polynomial* associated with the extension L/K and the system of difference generators η.

Note that the existence of Ehrhart-type dimension quasi-polynomials associated with weighted filtrations of differential and inversive difference modules was established by C. Dönch in his dissertation [1]. C. Dönch also proved the existence of dimension quasi-polynomials associated with finitely generated differential and inversive difference field extensions (see [1, Theorem 3.1.13]) using the technique of weight Gröbner bases in the associated modules of Kähler differentials. This method, however, cannot be applied to a non-inversive difference field extension, since in this case there is no natural difference module structure of the corresponding module of Kähler differentials. In order to prove the above theorem, we apply the technique of characteristic sets (with respect to a ranking that respects the weighted order of transforms) and the following proposition that generalizes Kolchin's result on dimension polynomials of subsets of $\mathbb{N}^m$ (see [2, Chapter 0, Lemma 16]).

Let $\leq_P$ denote the product order on $\mathbb{N}^m$ ($m \in \mathbb{N}$, $m \geq 1$), that is a partial order such that $(a_1, \dots, a_m) \leq_P (b_1, \dots, b_m)$ if and only if $a_i \leq b_i$ for $i = 1, \dots, m$. Let $w_1, \dots, w_m$ be fixed non-negative rational numbers and for any $a = (a_1, \dots, a_m) \in \mathbb{N}^m$, let $\text{ord}_{\,w} a = w_1 a_1 + \dots + w_m a_m$.

If $A \subseteq \mathbb{N}^m$ and $r \in \mathbb{N}$, then $A(r)$ will denote the set of all $a = (a_1, \dots, a_m) \in A$ such that $a_1 + \dots + a_m \leq r$. Furthermore, if $A \subseteq \mathbb{N}^m$, then V_A will denote the set of all m-tuples $v = (v_1, \dots, v_m) \in \mathbb{N}^m$ that are not greater than or equal to any m-tuple from A with respect to $\leq_P$. (Clearly, an element $v = (v_1, \dots, v_m) \in \mathbb{N}^m$ belongs to V_A if and only if for any element $(a_1, \dots, a_m) \in A$ there exists $i \in \mathbb{N}, 1 \leq i \leq m$, such that $a_i > v_i$.)

Proposition. *With the above conventions, for any set $A \subseteq \mathbb{N}^m$, there exists a quasi-polynomial $\omega_A(t)$ in one variable t such that*

(i) $\omega_A(r) = \text{Card}\, V_A(r)$ *for all sufficiently large* $r \in \mathbb{N}$.

(ii) $\deg \omega_A \leq m$.

(iii) $\deg \omega_A = m$ *if and only if* $A = \emptyset$.

(iv) $\omega_A = 0$ *if and only if* $(0, \dots, 0) \in A$.

With the notation of the theorem, one can use the above proposition to express the quasi-polynomial $\Phi_{\eta|K}(t)$ as an alternative sum of quasi-polynomials of the form $\omega_A(t)$ where $A \subseteq \mathbb{N}^m$. It allows one to obtain methods of computation of difference dimension quasi-polynomials based on known algorithms for computing Ehrhart quasi-polynomials of rational polytopes. We will give some corresponding examples and discuss an interpretation of a difference dimension quasi-polynomial as the strength of a system of difference equations with weighted translations. Such systems arise, in particular, from finite difference approximations of systems of PDEs with weighted derivations (see, for example, [5] and [6]).

References

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