Gröbner basis driven construction of a new s-consistent difference approximation to Navier-Stokes equations

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The quality of a numerical solution to a partial differential equation (PDE) or to a system of PDEs obtained by the finite difference method is determined by the underlying finite difference approximation (FDA) to the PDE(s). It is a challenging problem to construct a FDA which inherits or mimics at the discrete level the fundamental properties of the original PDE(s) such as topology, symmetries, conservation laws, maximum principle, etc. Such FDAs are called *compatible* or *mimetic* (cf. [2]).

In [7], for linear PDE systems and regular (Cartesian) grids, the necessary condition for compatibility of FDA, *s(strong)-consistency*, was established. This condition admits an algorithmic verification via difference Gröbner bases. It was generalized in [3, 6, 8] to polynomially-nonlinear systems of PDEs.

Let F be a finite set of differential polynomials, $\{f = 0 \mid f \in F\}$ be the corresponding PDE system, $\tilde{F}$ be the set of difference polynomials such that set $\{\tilde{f} = 0 \mid \tilde{f} \in \tilde{F}\}$ forms a FDA to the PDE system on a chosen (regular) solution grid. Then FDA is called s-consistent if

$$(\forall \tilde{g} \in [\![\tilde{F}]\!])\,(\exists g \in [\![F]\!])\,[\tilde{g} \text{ is FDA to } g]$$

where $[\![\tilde{F}]\!]$ and $[\![F]\!]$ is respectively the perfect difference ideal (cf. [9]) generated by $\tilde{F}$ and the perfect differential ideal generated by $\tilde{F}$.

By applying the approach of paper [5] to the two-dimensional Navier-Stokes equations describing the unsteady motion of an incompressible viscous liquid of constant viscosity

$$\begin{cases} f_1 := & u_x + v_y = 0\,, \\ f_2 := & u_t + uu_x + vu_y + p_x - \frac{1}{\mathrm{Re}}\,\Delta u = 0\,, \\ f_3 := & v_t + uv_x + vv_y + p_y - \frac{1}{\mathrm{Re}}\,\Delta v = 0\,. \end{cases} \tag{1}$$

where (u,v) is the velocity field, p is the pressure, the constant Re is the Reynolds number, we constructed in [4] two s-consistent FDAs to (1). Below we refer to these FDAs as to FDA2 and FDA3. In [1] we showed the numerical superiority of FDA2 over two other FDAs that are not s-consistent. In doing so, we used the exact solution [10] to (1)

$$u=-e^{-\frac{2t}{\mathrm{Re}}}\cos(x)\sin(y),\ v=e^{-\frac{2t}{\mathrm{Re}}}\sin(x)\cos(y),\ p=-\frac{1}{4}e^{-\frac{4t}{\mathrm{Re}}}(\cos(2x)+\cos(2y)). \quad (2)$$

and the grid with temporal spacing τ and the spatial spacing h for both x and y.

In this talk we propose a new FDA that we obtained in the following way. We have started with a direct difference approximation of the single equations (1). In the conventional notations

$$u^n_{j,k} := u(n\tau, xj, yk), \quad v^n_{j,k} := v(n\tau, xj, yk), \quad p^n_{j,k} := p(n\tau, xj, yk)$$

for the grid functions u, v, p such approximation reads

$$\begin{aligned}
&\frac{u^n_{j+1,k}-u^n_{j-1,k}}{2h}+\frac{v^n_{j,k+1}-v^n_{j,k-1}}{2h}=0,\\
&\frac{u^{n+1}_{j,k}-u^n_{j,k}}{\tau}+\frac{(u^n_{j+1,k})^2-(u^n_{j-1,k})^2}{2h}+\frac{v^n_{j,k+1}u^n_{j,k+1}-v^n_{j,k-1}u^n_{j,k-1}}{2h}+\frac{p^n_{j+1,k}-p^n_{j-1,k}}{2h}\\
&\qquad -\frac{1}{\mathrm{Re}}\left(\frac{u^n_{j+1,k}-2u^n_{j,k}+u^n_{j-1,k}}{h^2}+\frac{u^n_{j,k+1}-2u^n_{j,k}+u^n_{j,k-1}}{h^2}\right)=0, \qquad (3)\\
&\frac{v^{n+1}_{j,k}-v^n_{j,k}}{\tau}+\frac{(v^n_{j,k+1})^2-(v^n_{j,k-1})^2}{2h}+\frac{u^n_{j+1,k}v^n_{j+1,k}-u^n_{j-1,k}v^n_{j-1,k}}{2h}+\frac{p^n_{j,k+1}-p^n_{j,k-1}}{2h}\\
&\qquad -\frac{1}{\mathrm{Re}}\left(\frac{v^n_{j+1,k}-2v^n_{j,k}+v^n_{j-1,k}}{h^2}+\frac{v^n_{j,k+1}-2v^n_{j,k}+v^n_{j,k-1}}{h^2}\right)=0.
\end{aligned}$$

Aiming to obtain a time-independent equation with linear leading monomial in the variable p in order to solve numerically the FDA, we have performed a difference Gröbner basis computation with pure lexicographic ordering with $p > u > v$ and $\partial_t > \partial_x > \partial_y$. We have obtained then a finite difference Gröbner basis consisting of 5 elements where we have found an equation of the required form, namely

$$\begin{aligned}
&\frac{p^n_{j+2,k}-2p^n_{j,k}+p^n_{j-2,k}}{4h^2}+\frac{p^n_{j,k+2}-2p^n_{j,k}+p^n_{j,k-2}}{4h^2}\\
&+\frac{(u^n_{j+2,k})^2-2(u^n_{j,k})^2+(u^n_{j-2,k})^2}{4h^2}+\frac{(v^n_{j,k+2})^2-2(v^n_{j,k})^2+(v^n_{j,k-2})^2}{4h^2}\\
&+2\frac{u^n_{j+1,k+1}v^n_{j+1,k+1}-u^n_{j+1,k-1}v^n_{j+1,k-1}-u^n_{j-1,k+1}v^n_{j-1,k+1}+u^n_{j-1,k-1}v^n_{j-1,k-1}}{4h^2} \qquad (4)\\
&+\frac{2}{\mathrm{Re}}\frac{-u^n_{j+2,k}+4u^n_{j+1,k}-4u^n_{j-1,k}+u^n_{j-2,k}-u^n_{j+1,k+1}-u^n_{j+1,k-1}+u^n_{j-1,k+1}+u^n_{j-1,k-1}}{4h^3}\\
&+\frac{2}{\mathrm{Re}}\frac{-v^n_{j,k+2}+4v^n_{j,k+1}-4v^n_{j,k-1}+v^n_{j,k-2}-v^n_{j+1,k+1}-v^n_{j-1,k+1}+v^n_{j+1,k-1}+v^n_{j-1,k-1}}{4h^3}=0.
\end{aligned}$$

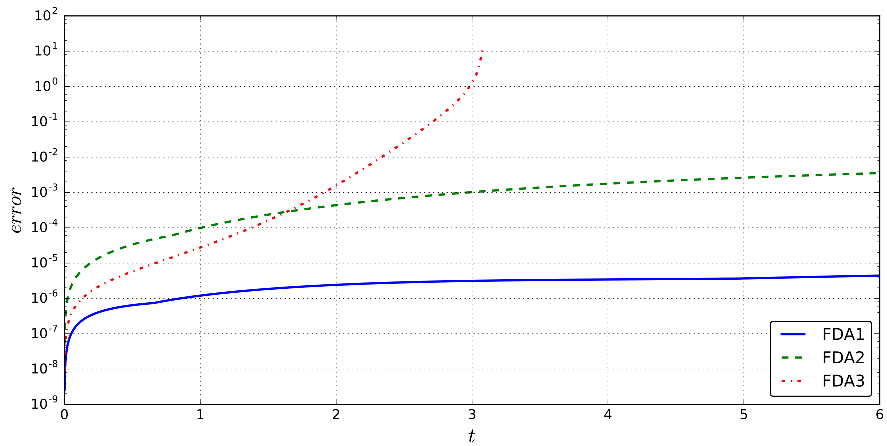

Figure 1: Dynamics of numerical error

It is interesting to note that this computer-generated difference equation is in fact the approximation of the following differential equation

$$(p_{xx}+p_{yy})+2(u_x^2+u_xv_y+u_yv_x+v_y^2+u(u_{xx}+v_{xy})+v(u_{xy}+v_{yy}))+ \\ -\frac{1}{\mathrm{Re}}(u_{xxx}+u_{xyy}+v_{xxy}+6v_{yyy})=0.$$

One can check that this equation belongs to the differential ideal generated by the Navier-Stokes equations (1) which provides the s-consistency of a new scheme FDA1 that we have obtained by joining the equation (4) to the difference system (3).

Figure 1 demonstrates dynamics of the numerical errors in computation of pressure p for the three difference approximations, FDA1 and two s-consistent FDAs (FDA2 and FDA3) constructed in [4], and for the Cauchy problem with initial data taken according to (2). As one can see from the figure, the numerical behavior of FDA1 is much better than that of FDA2 and FDA3. In doing so, FDA3 reveals instability. The errors in u and v have similar behavior.

References

[1] P. Amodio, Yu.A. Blinkov, V. P. Gerdt, R. La Scala. On Consistency of Finite Difference Approximations to the Navier-Stokes Equations. *Computer Algebra in Scientific Computing / CASC 2013*, V. P. Gerdt, W. Koepff, E. W. Mayr, E. V. Vorozhtsov (Eds.), LNCS 8136, pp. 46–60. Springer, Cham, 2013. arXiv:math.NA/1307.0914

[2] D. N. Arnold, P. B. Bochev, R.B. Lehoucq, R. A.Nicolaides, M. Shashkov (Eds.) Compatible Spatial Discretizations. *The IMA Volumes in Mathematics and its Applications*, Vol. 142. Springer, 2006.

[3] V. P. Gerdt. Consistency Analysis of Finite Difference Approximations to PDE Systems. Mathematical *Modelling in Computational Physics / MMCP 2011*, Adam, G., Buša, J., Hnatič, M. (Eds.), LNCS 7125, pp. 28–42. Springer, Berlin (2012). arXiv:math.AP/1107.4269

[4] V. P. Gerdt, Yu.A. Blinkov. Involution and Difference Schemes for the Navier-Stokes Equations. *Computer Algebra in Scientific Computing / CASC 2009*, V. P. Gerdt, E. W. Mayr, E. V. Vorozhtsov (Eds.), LNCS 5743, pp. 94–105. Springer, Berlin, 2009.

[5] V. P. Gerdt, Yu. A. Blinkov, V. V. Mozzhilkin. Gröbner Bases and Generation of Difference Schemes for Partial Differential Equations. *Symmetry, Integrability and Geometry: Methods and Applications*, 2:26, 2006.

[6] V. P. Gerdt, R. La Scala, Noetherian quotients of the algebra of partial difference polynomials and Gröbner bases of symmetric ideals, *Journal of Algebra*, 423, 1233–1261, 2015.

[7] V. P. Gerdt, D. Robertz. Consistency of Finite Difference Approximations for Linear PDE Systems and its Algorithmic Verification). Watt, S.M. (Ed.), *Proceedings of ISSAC 2010*, pp. 53–59. Association for Computing Machinery (2010).

[8] R. La Scala, Gröbner bases and gradings for partial difference ideals, *Math. Comp.*, 84, 959–985, 2015.

[9] A. Levin. Difference Algebra. *Algebra and Applications*, Vol. 8. Springer, 2008.

[10] J. Kim, P. Moin. Application of a Fractional-Step Method To Imcompressible Navier-Stokes Equations. *J. Comput. Phys.* 59, 308–323, 1985.